**Use of Probability Density Functions to Predict Diameter Distribitions in Forestry**

 ***Muammer ŞENYURT1,[[1]](#footnote-1)\*, İlker ERCANLI3***

*1 Forest Faculty, Forest Engineering Department, Çankırı Karatekin University, Çankırı, TURKIYE*

|  |
| --- |
| **Abstract** This study discusses the role of diameter distribution models in providing more detailed insights into forest stand structures, essential for sustainable forest management and product determination. Traditional yield tables used in forestry provide stand-level predictions for growth and yield but lack specificity at finer scales like diameter classes. The paper highlights that, especially in Turkey, inventory studies for forest planning do not produce data detailed enough to fully support sustainable forestry practices. Diameter distribution models, using probability density functions, allow for the estimation of stand components, such as tree count, basal area, and volume, by diameter classes. By integrating stand models with diameter distribution models, it becomes possible to derive more granular predictions that can benefit forestry applications, including forest management and silviculture. These models enable enhanced accuracy in predicting stand structure variations, which in turn supports planning for a broader array of forest products. The study emphasizes that the implementation of diameter distribution models can help mitigate current limitations in sustainable forestry efforts by enabling data from inventory studies to be analyzed at the diameter step level. As a result, this modeling approach offers a valuable foundation for forest management activities, aligning forestry practices with sustainability goals. |
| Keywords: Probability density function, Diameter distrubition, model |

1. **Introduction**

The forest is a life community formed by a combination of forest soil and an array of vegetation, including trees, shrubs, herbaceous plants, mosses, ferns, and fungi of specific heights, structures, and densities, which collectively establish a unique climate over a broad area. This ecosystem is further enriched by microorganisms living both above and below the soil, as well as various insects and animals (Aytuğ, 1976). Forests serve as habitats not only for humankind but for all terrestrial living beings. In this regard, making optimal use of the services and values forests provide, while ensuring the integrity of the ecosystem and the sustainability of these areas, is achievable through their planned management. The consistency of such plans and the selection of an effective and systematic management approach depend on the quantification of the products and services offered by forests, as well as the modeling of the data obtained (Başkent & Keleş, 2004; Bolat, 2014).

Information about the increment and growth potential of forests, as well as stand structures, and the relationships between stand structures and site productivity, age, density, and species composition, is crucial for forest planning (Yavuz et al., 2002). Stand structures are defined by the distribution of trees within diameter classes in a stand, and the current and future diameter distributions provide a critical knowledge base, especially for forest management practices (Maltoma, 1997). Determining the variety of products that can be derived from forests depends on estimating the diameter distributions of stands, and this information is essential for effective forest planning (Rennols et al., 1985; Borders & Patterson, 1990; Laar & Akça, 2007).

Vanclay (1994), as well as Gadow and Hui (1999), defined models in the context of forestry science as systems of equations that estimate the growth and yield values of stands under various conditions. Burkhart (1995) and Garcia (2001) further described the applications of these estimates in forestry, including the evaluation of silvicultural treatment options, updating inventory data, and determining the timber yield obtainable from stands.

In the development process of models, which have a history of over 200 years in forestry, there are two primary types: empirical growth models and process-based (ecologically-based) growth models (Porté & Bartelink, 2002). Empirical growth models predict growth and yield elements for individual trees or stands using statistical functions (allometric relationships) that incorporate various tree and stand variables (Burkhart, 1997). Based on the unit used in modeling, empirical models are categorized into three types: Whole-Stand Models, Size-Class Models, and Individual-Tree Models (Mısır, 2003).

Diameter distribution models, which are part of size-class models, are widely used in forestry at the stand level. Compared to the other two model classes (whole-stand and individual-tree models), diameter distribution models offer several advantages: they provide insights into stand structure, include information at the diameter class level, and offer data that can be readily obtained from general forestry inventories. These models also provide information on the variety of products that can be derived from a stand, making them valuable for detailed stand-level information in forestry.

The need for predictions at the diameter class or diameter step level, rather than just at the stand level, has driven extensive research on diameter distribution models. Early studies include Gram’s work in 1883, which modeled the diameter distribution of beech stands using a Normal distribution, and De Liocourt’s 1898 study, which applied an Exponential distribution to the diameter distributions of uneven-aged stands. However, interest in modeling diameter distributions intensified particularly in the 1960s, when probability density functions (pdfs), a key concept in statistics, began to be applied for modeling diameter distributions in forestry (Packard, 2000).

This study provides information on probability density functions used in diameter distribution modeling, a significant area in forestry. Additionally, it briefly outlines the evaluation processes used to assess the effectiveness of these functions in modeling diameter distributions.

1. **Probability Density Functions**

Probability density functions are statistical functions that estimate the frequency value corresponding to a specific observed diameter as a ratio of the total number of individuals in the population from which the data were measured, yielding estimates between 0 and 1 (Bailey & Dell, 1973). Various researchers have developed different methods for modeling diameter distributions. For instance, the Johnson SB function was introduced by Johnson (1949), the Weibull function by Weibull (1951), the Gamma function by Nelson (1964), the Log-normal function by Bliss and Reineker (1964), and the Beta function by Clutter and Bennet (1965). Numerous studies have since used these functions to model diameter distributions (Bailey & Dell, 1973; Smalley & Bailey, 1974; Haffley & Schreuder, 1977; Rennols et al., 1985; Knoebel et al., 1986; Pukkala et al., 1990; Saramaki, 1992; Maltamo et al., 1995; Maltamo, 1997; Packard, 2000; Liu et al., 2004; Palahi et al., 2006; Podlaski, 2006; Nord-Larsen & Cao, 2006; Palahi et al., 2007).

In forestry, various probability density functions are used to model the distribution of trees within diameter classes in a stand, including the Normal (Bailey, 1980), Lognormal (Bliss & Reinker, 1964), Gamma (Nelson, 1964), Beta (Clutter & Bennet, 1965; Zöhrer, 1969), Johnson’s SB (Johnson, 1949), and Weibull distributions (Weibull, 1951; Bailey & Dell, 1973) (Ercanlı & Yavuz, 2010). However, recent studies have highlighted the prominence of the 3-parameter Weibull distribution and 4-parameters Johnson SP. These probability density functions, with its flexible 3 and 4 parameters structure, have proven particularly effective in modeling diverse diameter distributions.

In addition to the traditional and widely recognized probability density functions (PDFs) such as the Gamma, Beta, Weibull, and Johnson’s SB functions, statistical science has introduced other distribution functions, including the Laplace, Rayleigh, Nakagami, Lévy, Rice, and Kumaraswamy distributions (Michalowicz et al., 2013). The Rice distribution, applicable to positive real numbers, is closely related to several well-known distributions such as Chi-Square, Normal, Log-Normal, and Rayleigh (Jiang et al., 2018). The Rayleigh distribution is a specific case of the 2-parameter Weibull distribution, named after the English physicist Lord Rayleigh (Aslam et al., 2015). Emerging in 1960, the Nakagami distribution is relatively recent and is widely applied to model right-skewed, positive data sets (Akgül & Şenoğlu, 2023). The Lévy distribution is notable for its continuous, stable properties for non-negative random variables (Knopova & Schilling, 2013; Yousof et al., 2022). The Laplace distribution, one of the oldest known distributions, is unimodal and symmetric with a sharper peak than the Normal distribution (Liu & Kozubowski, 2015). Lastly, Kumaraswamy's distribution shares many properties with the Beta distribution but offers advantages in tractability and applicability to a range of natural phenomena, as it is a versatile PDF for double-bounded random processes (El-Sagheer, 2019). The structural models of the probability density functions mentioned are presented in Table 1 (Şahin and Ercanlı, 2023).

**Table 1.** Various PDFs for modelling diameter distributions (Şahin and Ercanlı, 2023)

|  |  |  |  |
| --- | --- | --- | --- |
| **No.** | **Distribution** | **Density function** | **Parameters** |
| 1 | Gamma (3P) |  | *α*: continuous shape parameter ( *> 0*)*β*: continuous scale parameter ( *> 0*)*γ*: continuous location parameter: Gamma function*γ* ≤ *x* < +*ꝏ* |
| 2 | Johnson’s SB (4P) |  | *γ*, *δ*:continuous shape parameters(*δ* > 0)*λ*: continuous scale parameter (*λ*>0)*ξ*: continuous location parameter*ξ ≤ x ≤ ξ+ λ* |
| 3 | Kumaraswamy (4P) |  | *α1*, *α2*: continuous shape parameters (*α1*, *α2* > 0)*a, b*: continuous boundary parameters (*a < b*)*a ≤ x ≤ b* |
| 4 | Laplace (2P) |  | *λ*: continuous inverse scale parameter (*λ*>0)*μ*: continuous location parameter- *ꝏ* < x < + *ꝏ* |
| 5 | Lévy (1P) |  | *σ*: continuous scale parameter (σ > 0) |
| 6 | Lévy (2P) |  | *γ*: continuous location parameter (yields the one-parameter Lévy distribution)*γ* < x < + *ꝏ* |
| 7 | Lognormal (2P) |  | σ and µ: continuous parameters (σ > 0) |
| 8 | Lognormal (3P) |  | γ: continuous location parameter (yields the two-parameter Lognormal distribution)*γ* < x < + *ꝏ* |
| 9 | Nakagami (2P) |  | *m*: continuous parameter (*m* ≥ 0.5)*Ω*: continuous parameter (Ω > 0) *0 ≤ x <* + *ꝏ* |
| 10 | Normal (2P) |  | *σ*: continuous scale parameter (σ > 0)*µ*: continuous location parameter- *ꝏ* < x < + *ꝏ* |
| 11 | Rayleigh (1P) |  | *σ*: continuous scale parameter (σ > 0) |
| 12 | Rice (2P) |  | *v*, *σ*:continuous parameters(*v* ≥ 0; σ > 0)*I0*: modified Bessel function of the first kind of zero*0 ≤ x <* + *ꝏ* |
| 13 | Weibull (2P) |  | : continuous shape parameter ( *> 0*): continuous scale parameter ( *> 0*) |
| 14 | Weibull (3P) |  | γ: continuous location parameter (yields the two-parameter Weibull distribution)*γ* ≤ x < + *ꝏ* |

\* x is the tree diameter here

In forestry, various methods are used to estimate the parameters of the probality density functions (given in Table 1), which is widely applied in modeling diameter distributions. These methods include: (i) Nonlinear Regression Analysis, (ii) Maximum Likelihood Estimation, (iii) Moment-Based Parameter Recovery, and (iv) Percentile-Based Parameter Recovery. In this study, percentile-based equations, which provide a simple, practical, and effective approach, and the maximum likelihood method were employed for modeling diameter distributions (Knowe 1992, Bailey et al. 1989, Knowe et al. 1997, Liu et al. 2004, Cao 2004, Poduel 2011, Poduel and Cao 2013). The most used method, Maximum Likelihood Estimation (MLE), is based on the similarity between diameter classes obtained from measured diameter values in the field and those estimated by the model (Bailey and Dell 1973, Borders et al. 1987). Estimating the parameters of probability density functions through MLE requires numerical solution techniques involving various iterations (Harter and Moore 1965). To apply MLE and estimate parameters, the likelihood function must be maximized. Statistical software such as SPSS, SAS, and R are commonly used for this purpose.

1. **Conclusion**

Diameter distribution models are essential for obtaining more detailed predictions about stand structures and determining the variety of products that can be derived from forests. In our country, yield tables based on normal and density-dependent estimates are currently used for predicting growth and yield, although these tables provide estimations for the stand as a whole. However, the need for more detailed predictions regarding stand structures is becoming increasingly apparent, especially in forest management and various forestry activities. Probability density functions and density-dependent yield tables allow for the estimation of the distribution of stand components—such as tree count, basal area, and volume—across diameter classes. By integrating stand and diameter distribution models, more detailed predictions can thus be obtained. Once the distribution of tree counts by diameter classes is achieved, the distributions of basal area and volume can also be derived using the tree count distributions. Diameter distribution models enable more detailed estimations of stand components, initially obtained for the entire stand, to be achieved at the level of diameter classes. These predictions, obtained with greater detail at the diameter class scale, form a valuable basis for various forestry applications, especially for forest management and silviculture.

Diameter distribution models enable predictions (such as stand volume, basal area, and tree count) provided by stand models and yield tables, typically at the whole-stand level, to be obtained in greater detail at the diameter class and step levels. This approach facilitates more precise predictions about stand structures and allows for a more accurate determination of the types of products that can be obtained from forests. In our country, Normal Yield Tables and Density-Dependent Yield Tables are used to predict growth and yield, providing estimations for the stand as a whole. However, the need for more detailed stand structure predictions is increasingly felt, especially in forest management and other forestry activities. The lack of detailed data obtained from inventory studies conducted for planning purposes in our country presents constraints on sustainable forestry practices. By enabling data obtained from inventory studies to be estimated at the diameter step level, diameter distribution models help alleviate some of these limitations in sustainable forestry efforts.

**References**

Akgül, F. G., & Şenoğlu, B. (2023). Comparison of wind speed distributions: a case study for Aegean coast of Turkey. *Energy Sources, Part A: Recovery, Utilization, and Environmental Effects*, *45*(1), 2453-2470.

Aslam, M., Tahir, M., Hussain, Z., & Al-Zahrani, B. (2015). A 3-component mixture of Rayleigh distributions: properties and estimation in Bayesian framework. *PloS one*, *10*(5), e0126183.

Aytuğ, B. (1976). Orman tanımlaması ve bu tanımlamada yer alan ağaç, ağaççık ve çalı kavramları. *I. Orman Kadastro Semineri*, OGM Yayın No: 607/13, Ankara

Bailey, R. L., & Dell, T. R. (1973). Quantifying diameter distributions with the Weibull function. *Forest science*, *19*(2), 97-104.

Bailey, R. L. (1980). Individual tree growth derived from diameter distribution models. *Forest Science*, *26*(4), 626-632.

Bailey, R. L., Burgan, T. M., & Jokela, E. J. (1989). Fertilized midrotation-aged slash pine plantations—stand structure and yield prediction models. *Southern Journal of Applied Forestry*, *13*(2), 76-80.

Başkent, E. Z., & Keleş, S., (2004). Ormancılıkta Model ve Modelleme Kavramlarının Kullanımı ve Genel Değerlendirmesi, *Orman Mühendisliği Dergisi* , vol.41, 19-24.

Bliss, C. I., & Reinker, K. A. (1964). A lognormal approach to diameter distributions in even-aged stands. *Forest Science*, *10*(3), 350-360.

Borders, B.E., Souter, R.A., Bailey, R.L. & Ware, K.D. 1987. Percentile based distributions characterize forest tables. *Forest Science*, 33(2); 570-576.

Burkhart, H. 1995. Modeling forest growth. *Encyclopedia of Environmental Biology*, (2); 535-543.

Burkhart, H. E. (1997). Development of empirical growth and yield models. *Empirical and process-based models for forest tree and stand growth simulation. Lisboa: Salamandra*, 53-60.

Cao, Q. V. (2004). Predicting parameters of a Weibull function for modeling diameter distribution. *Forest science*, *50*(5), 682-685.

Clutter, J. L., & Bennett, F. A. (1965). Diameter distributions in old-field slash pine plantations.

EL-Sagheer, R. M. (2019). Estimating the parameters of Kumaraswamy distribution using progressively censored data. *Journal of Testing and Evaluation*, *47*(2), 905-926.

Ercanlı, İ., & Yavuz, H. (2010). Doğu ladini (Picea Orientalis (L.) Link)-Sarıçam (Pinus Sylvestris L.) karışık meşcerelerinde çap dağılımlarının olasılık yoğunluk fonksiyonları ile belirlenmesi. *Kastamonu University Journal of Forestry Faculty*, *10*(1), 68-83.

Gadow, K.V. & Hui, G.Y. 1999. Modeling forest development. *Kluwer Academic Publishers*, 213 p., Netherlands.

Garcıa, O. (2001). Growth & Yield in British Columbia Background and discussion.

Hafley, W. L., & Schreuder, H. T. (1977). Statistical distributions for fitting diameter and height data in even-aged stands. *Canadian Journal of Forest Research*, *7*(3), 481-487.

Johnson, N. L. (1949). Systems of frequency curves generated by methods of translation. *Biometrika*, *36*(1/2), 149-176.

Jiang, K., Chen, X., Zhu, Q., Chen, L., Xu, D., & Chen, B. (2018). A novel simulation model for nonstationary rice fading channels. *Wireless Communications and Mobile Computing*, *2018*(1), 8086073.

Knoebel, B. R., Burkhart, H. E., & Beck, D. E. (1986). A growth and yield model for thinned stands of yellow-poplar. *Forest Science*, *32*(suppl\_2), a0001-z0002.

Knopova, V., & Schilling, R. L. (2013, January). A note on the existence of transition probability densities of Lévy processes. In *Forum Mathematicum* (Vol. 25, No. 1, pp. 125-149). Walter de Gruyter GmbH.

Knowe, S. A. (1992). Basal area and diameter distribution models for loblolly pine plantations with hardwood competition in the Piedmont and Upper Coastal Plain. *Southern Journal of Applied Forestry*, *16*(2), 93-98.

Knowe, S. A., Ahrens, G. R., & DeBell, D. S. (1997). Comparison of diameter-distribution-prediction, stand-table-projection, and individual-tree-growth modeling approaches for young red alder plantations. *Forest Ecology and Management*, *98*(1), 49-60.

Liu, C., Zhang, S. Y., Lei, Y., Newton, P. F., & Zhang, L. (2004). Evaluation of three methods for predicting diameter distributions of black spruce (Picea mariana) plantations in central Canada. *Canadian Journal of Forest Research*, *34*(12), 2424-2432.

Liu, Y., & Kozubowski, T. J. (2015). A folded Laplace distribution. *Journal of Statistical Distributions and Applications*, *2*, 1-17.

Maltamo, M., Puumalainen, J., & Päivinen, R. (1995). Comparison of beta and Weibull functions for modelling basal area diameter distribution in stands of Pinus sylvestris and Picea abies. *Scandinavian Journal of Forest Research*, *10*(1-4), 284-295.

Maltamo, M. 1997. Comparing basal area diameter distributions estimated by tree species and for the entire growing stocks in mixed stand. *Silva Fennica*, 31(1); 53-65.

Michalowicz, J. V., Nichols, J. M., & Bucholtz, F. (2013). *Handbook of differential entropy*. Crc Press.

Misir, N. (2003). *Karaçam ağaçlandırmalarına ilişkin büyüme modelleri* (Doctoral dissertation, Doktora Tezi, Karadeniz Teknik Üniversitesi, Fen Bilimleri Enstitüsü, Trabzon).

Nelson, T. C. (1964). Diameter distribution and growth of loblolly pine. *Forest Science*, *10*(1), 105-114.

Nord-Larsen, T., & Cao, Q. V. (2006). A diameter distribution model for even-aged beech in Denmark. *Forest ecology and management*, *231*(1-3), 218-225.

Packard, K. C. (2000). *Modeling tree diameter distributions for mixed-species conifer forests in the Northeast United States*. State University of New York College of Environmental Science and Forestry.

Palahí, M., Pukkala, T., & Trasobares, A. (2007). Calibrating predicted tree diameter distributions in Catalonia, Spain. *Silva Fennica*, *40*(3), 487.

Podlaski, R. (2006). Suitability of the selected statistical distributions for fitting diameter data in distinguished development stages and phases of near-natural mixed forests in the Świętokrzyski National Park (Poland). *Forest Ecology and Management*, *236*(2-3), 393-402.

Poudel, K. P., & Cao, Q. V. (2013). Evaluation of methods to predict Weibull parameters for characterizing diameter distributions. *Forest Science*, *59*(2), 243-252.

Porte, A., & Bartelink, H. H. (2002). Modelling mixed forest growth: a review of models for forest management. *Ecological modelling*, *150*(1-2), 141-188.

Pukkala, T., Saramäki, J., & Mubita, O. (1990). Management planning system for tree plantations. A case study for Pinus kesiya. *Silva Fennica*, *24*(2), 171-180.

Rennolls, K., Geary, D.N. & Rollinson, T.J.D. 1985. Characterizing diameter distributions by the use of the Weibull distributions. *Forestry,* 58(1); 57-66.

Samaraki, J. (1992). A growth and yield prediction model of pinus kesiya in zambia. *Acta Forestalia Fennica*, *230*, 68.

Smalley, G. W. (1974). *Yield tables and stand structure for shortleaf pine plantations in Tennessee, Alabama, and Georgia highlands* (Vol. 97). Southern Forest Experiment Station, Forest Service, US Department of Agriculture.

Şahin, A., & Ercanli, I. (2023). An evaluation of various probability density functions for predicting diameter distributions in pure and mixed-species stands in Türkiye. *Forest systems*, *32*(3), 2.

Van Laar, A. & Akça, A. (2007). *Forest mensuration* (Vol. 13). Springer Science & Business Media.

Vanclay, J.K. 1994. Modelling forest growth: Applications to mixed tropical forests. *CAB International, Department of Economics and Natural Resource*, Royal Veterinary and Agricultural University, 312 p., Copenhagen, Denmark.

Weibull, W. (1951). A statistical distribution function of wide applicability. *Journal of applied mechanics*.

Yavuz, H., Gül, A.U., Mısır, N., Özçelik, R. & Sakıcı, O.E. 2002. Meşcerelerde çap dağılımının düzenlenmesi ve bu dağılımlara ilişkin parametreler ile çeşitli meşcere öğeleri arasındaki ilişkilerin belirlenmesi*. Orman Amenajmanı’nda Yeni Kavramsal Açılımlar ve Yeni Hedefler Sempozyumu*, 2012, İstanbul.

Yousof, H. M., Korkmaz, M. C., Hamedani, G. G., & Ibrahim, M. (2022). A novel Chen extension: Theory, characterizations and different estimation methods. *European Journal of Statistics*, *2*, 1-1.

Zöhrer, F. (1969). Ausgleich von Häufigkeitsverteilungen mit hilfe der beta-funktion. *Forstarchiv*, *40*(3), 37-42.

1. \* Corresponding author. *e-mail address:msenyurt@karatekin.edu.tr* [↑](#footnote-ref-1)